

# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 1895

EFFECT OF ASPECT RATIO ON UNDAMPED TORSIONAL  
OSCILLATIONS OF A THIN RECTANGULAR

WING IN SUPERSONIC FLOW

By Charles E. Watkins

Langley Aeronautical Laboratory  
Langley Air Force Base, Va.



Reproduced From  
Best Available Copy

Washington

June 1949

**DISTRIBUTION STATEMENT A**  
Approved for Public Release  
Distribution Unlimited

20000731 182

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

---

TECHNICAL NOTE 1895

---

EFFECT OF ASPECT RATIO ON UNDAMPED TORSIONAL  
OSCILLATIONS OF A THIN RECTANGULAR

WING IN SUPERSONIC FLOW

By Charles E. Watkins

SUMMARY

The theory for single-degree torsional instability of a two-dimensional wing oscillating in a supersonic stream is extended so as to apply to a finite rectangular wing oscillating in a supersonic stream. The velocity potential and aerodynamic-torsional-moment coefficient based on the linearized equations of motion for small disturbances are derived by means of appropriate distributions of moving sources and doublets. The aerodynamic-torsional-moment coefficient thus derived is combined with a mechanical-damping coefficient to study the effect of aspect ratio on the undamped torsional oscillations of a finite rectangular wing. Decreasing the aspect ratio of the wing is found to have a highly stabilizing effect on the undamped torsional oscillations. Results of some selected calculations are presented in several figures.

It is pointed out that second-order thickness effects may be of significance.

INTRODUCTION

In theoretical studies of an oscillating wing in two-dimensional supersonic flow Possio noted, reference 1, that under certain conditions a single-degree torsional instability is possible. This instability, also sometimes known as a type of "single-degree flutter," was briefly discussed by Temple and Jahn in reference 2 and has since been further investigated by Garrick and Rubinow in reference 3, by W. P. Jones in reference 4, and by Cheilek and Frissel in reference 5.

It is pointed out in reference 3 that this single-degree flutter is due to the wing being negatively damped in torsion and that the negative damping is associated with a change in sign of the torsional-damping coefficient; furthermore in the two-dimensional case the instability may take place in a low supersonic Mach number

range ( $1 < \text{Mach number} < \sqrt{2.5}$ ), at low values of the frequency parameter, and for axis-of-rotation locations forward of a point two-thirds of the chord distance from the leading edge. This phenomenon may be of particular importance in connection with high-speed airplanes that are flown in (or through) the range of low supersonic Mach numbers at which the phenomenon may take place. Although in general discussions this phenomenon is usually associated with the word "wing" the theory applies as well to an aileron when the aileron is considered as a separate degree of freedom. Hereafter in this discussion the word "wing" may generally be given the broader interpretation of "wing or aileron."

The purpose of the present paper is to extend the theoretical investigation of the single-degree torsional instability of a wing oscillating in two-dimensional supersonic flow, as presented in reference 3, to a finite rectangular wing and to determine the effect of aspect ratio on the undamped torsional oscillations of such a wing. It is assumed that the negative damping phenomenon in three-dimensional flow, like that in two-dimensional flow, is determined mainly by low-order terms of a low frequency and that only the effect of the first power of the frequency must be considered to get a good approximation of its total effect.

In order to obtain the three-dimensional velocity potential and aerodynamic-damping coefficient the method suggested by Garrick and Rubinow in reference 6, which is briefly discussed in subsequent paragraphs, is applied to a thin, flat, rectangular wing performing slow, sinusoidal, torsional oscillations in a supersonic stream. The particular wing treated is such that the Mach cones emanating from the foremost point of each tip do not intersect the opposite tip ahead of the trailing edge of the wing.

The procedure developed herein may be readily extended to apply to any plan form with supersonic trailing edge as long as other edges that might be in the regions of mixed supersonic flow are continuously straight. What might be a more desirable extension would be to obtain the non-linear effect that thickness might have on the undamped torsional oscillations of a given plan form.

In reference 6, Garrick and Rubinow make use of the theory of small perturbations to investigate the air forces on a thin finite wing oscillating in a supersonic stream. For convenience the boundary-value problems for the velocity potential for a three-dimensional surface (finite wing) moving at supersonic speed are classified into two types and referred to as "purely supersonic" and "mixed supersonic." The purely supersonic boundary-value problem refers to regions of flow where no interaction between the flow on the upper and lower surfaces is present. In this case the surfaces can be treated separately and the boundary-value problem for each surface can be satisfied by source and sink distributions. The source and sink distributions for each surface

are known functions of the plan form and profile of the wing. Consequently, the velocity potentials in the purely supersonic region can always be expressed in the form of surface integrals with known integrands.

The mixed-supersonic boundary-value problem refers to regions where interaction between the flow on the upper and lower surfaces is present. This problem cannot be solved by a distribution of sources alone over the wing surface but in general can be satisfied by use of doublet distributions. The manner in which the doublets are to be distributed (or the distribution function) depends on the camber of the wing and on the plan form of the region of the wing where interaction between the flow on the upper and lower surfaces is present. In order to find a required distribution function it is usually possible to make use of the given boundary conditions and express the distribution function as the unknown function in an integral equation.

It may be appropriate to mention that Evvard, reference 7, has recently developed, by consideration of a source distribution over the entire upwash field, a time-dependent velocity potential that may be applied to certain edge problems. In reference 8 Harmon made use of this development to derive some stability derivatives for thin rectangular wings at supersonic speeds. The velocity potential derived herein for the slowly oscillating case can be shown to bear a relationship to the sum of three potentials employed by Harmon, namely those due to (1) constant vertical motion, (2) accelerated vertical motion, and (3) pitching motion.

#### SYMBOLS

$\phi$	disturbance-velocity potential
$\phi_1$	potential function of a moving source defined in equation (7)
$\phi_2$	potential function of a moving doublet defined in equation (16)
$\phi_N$	potential function due to distribution of sources in region N of figure 1
$\phi_T$	potential function due to distribution of doublets in region T of figure 1
x, y, z	rectangular coordinates attached to wing moving in negative x-direction

$Z_m$	function defining mean ordinates of any chordwise section of wing such as at $y = y_1$ as shown in figure 1
$w(x, y_1, t)$	vertical velocity at surface of wing along chordwise section at $y = y_1$ ; $\left( \left( \frac{\partial \phi}{\partial z} \right)_{z=0} \right)$
$\xi, \eta$	rectangular coordinates used to represent space location of sources or doublets in the $xy$ -plane
$x_0$	abscissa of axis of rotation of wing (elastic axis) as shown in figure 1
$t$	time
$\alpha$	angle of attack
$\dot{\alpha}$	time derivative of $\alpha$
$V$	velocity of main stream
$c$	velocity of sound
$M$	free-stream Mach number $(V/c)$
$\beta = \sqrt{M^2 - 1}$	
$\tau_1, \tau_2, \eta_1, \eta_2$	functions defined with equation (7)
$W(\xi, \eta)$	function used to represent space variation of source and doublet strengths
$w(t - \tau_1)$ $w(t - \tau_2)$ }	functions used to represent time variation of source and doublet strengths
$\omega$	angular frequency
$b$	one-half cord
$h$	one-half span
$A$	aspect ratio $\left( \frac{h}{b} \right)$

$F(\xi, \eta), f(\xi, \eta)$	functions used to denote doublet distributions
$\rho$	density
$p$	pressure difference, measured positive downwards, defined in equation (24)
$p_N, p_T$	values of $p$ in regions N and T, respectively, of figure 1
$M_\alpha$	aerodynamic moment, defined in equation (25)
$M_3$	in-phase component of aerodynamic-moment coefficient defined in equation (30)
$M_4$	out-of-phase component or (torsional-damping) aerodynamic-moment coefficient, defined in equation (31)
$M_4'$	total-torsional-damping coefficient
$g_\alpha$	mechanical-damping coefficient; $\pi g_\alpha$ corresponds to the usual logarithmic decrement
$\mu$	wing density parameter $\left( \frac{m}{4b^2} \right)$
$m$	mass of wing
$r_\alpha$	radius of gyration divided by $b$ ; $\left( \sqrt{\frac{I}{mb^2}} \right)$
$I$	moment of inertia per unit length of wing about elastic axis
$k = \frac{b\omega}{V}$	
$\frac{V}{b\omega_\alpha}$	flutter coefficient
$\omega_\alpha$	natural angular frequency of torsional vibration about elastic axis

### ANALYSIS

Consider a thin rectangular wing moving at a constant supersonic speed in a chordwise direction normal to its leading edge. In accordance with linear theory the boundary-value problem for the velocity potential is treated at the plane of the wing. For the portion of the wing between the Mach cones emanating from the foremost point of each tip, region N in figure 1(a), the boundary-value problem involves only the purely supersonic flow; and for the portions of the wing within these Mach cones, regions T in figure 1(a), the boundary-value problem involves the mixed supersonic flow. Thus the solution to the present problem proposes the use of the two types of boundary-value problems discussed in the introduction.

Boundary-value problems for velocity potential for a rectangular wing.—The differential equation for the propagation of small disturbances that must be satisfied by the velocity potential in both regions T and N (referred to a uniformly moving coordinate system as shown in fig. 1) is (equation 4, reference 6)

$$\frac{1}{c^2} \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right)^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \quad (1)$$

The boundary conditions that must be satisfied by the velocity potentials may be stated as follows: (a) In regions T and N the flow must at all times be tangent to the wing surface and (b) in regions T the pressure must fall to zero along the wing tips and remain zero in the portion of the Mach cones emanating from the foremost points of the wing tips not occupied by the wing. (As customary with linear boundary conditions the effects of thickness and camber are separated. In particular, in the nonstationary case the linear thickness effect is of no significance and is not considered here. The important camber effect is characterized by the conditions that the perturbation pressure, the perturbation velocity in the free-stream direction, and the perturbation-velocity potential are all antisymmetrical with respect to the reference plane, whereas the perturbation velocity normal to the reference plane is symmetrical. Accordingly, the boundary condition of zero pressure in the side wake may be stated as zero perturbation velocity in the free-stream direction. In view of these antisymmetric characteristics it is only necessary to derive the perturbation-velocity potentials for either the upper or lower surface of the wing. The upper surface is chosen for the derivations herein.)

With the boundary conditions stated as in the preceding paragraph, the difference in the two types of boundary-value problems involved

is in the additional conditions stipulated in condition (b) that must be satisfied in region T.

The condition of tangential flow may be expressed analytically as

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=0} = w(x, y_1, t) = V \frac{\partial Z_m}{\partial x} + \frac{\partial Z_m}{\partial t} \quad (2)$$

where  $Z_m$  is the function defining the mean ordinates at any chordwise section of the wing. For the particular case of the wing performing small sinusoidal oscillations of maximum amplitude  $\alpha_0$  about the spanwise axis  $x = x_0$ , the equation for  $Z_m$  may be written as (see fig. 1(b))

$$Z_m = \alpha(x - x_0) = \alpha_0 e^{i\omega t} (x - x_0) \quad (3)$$

Substituting this expression for  $Z_m$  into equation (2) gives for this case

$$w(x, t) = V\alpha + \dot{\alpha}(x - x_0) \quad (4)$$

Equation (4) indicates that the vertical motion of a wing moving forward and at the same time performing sinusoidal oscillations about a spanwise axis is equivalent to the superposition of the vertical motion of two similar wings moving forward, one at an instantaneous angle of attack and the other rotating about the spanwise axis  $x = x_0$  at an instantaneous rate of rotation. Since the differential equation (1) is linear, the velocity potential satisfying equations (1) and (4) may be considered as the sum of two potentials: the first corresponding to the first term of the right-hand side of equation (4) which hereinafter is denoted by  $\phi_\alpha$  and the second corresponding to the second term of the right-hand side of equation (4) which hereinafter is denoted by  $\phi_{\dot{\alpha}}$ . The symbolic forms of the velocity potentials for the regions N and T may therefore be written as

$$\phi_N = \phi_{N\alpha} + \phi_{N\dot{\alpha}} \quad (5)$$

and

$$\phi_T = \phi_{T\alpha} + \phi_{T\dot{\alpha}} \quad (6)$$

Derivation of  $\phi_N$ .—Reference 6 shows that the boundary-value problems for unsteady motion in the purely supersonic region (region N) may be satisfied by distributions of moving sources. The potential at any point  $(x, y, z)$  due to a moving source of strength  $w(t)$ , varying only with time, located at point  $(\xi, \eta, 0)$  is given in equation (7a) of reference 6. In the present notation the expression for this potential may be written as

$$\phi_1 = \frac{w(t - \tau_1) + w(t - \tau_2)}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} \quad (7)$$

where

$$\tau_1 = \frac{M(x - \xi)}{c\beta^2} - \frac{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\beta c}$$

$$\tau_2 = \frac{M(x - \xi)}{c\beta^2} + \frac{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\beta c}$$

$$\eta_1 = y - \frac{1}{\beta} \sqrt{(x - \xi)^2 - \beta^2 z^2}$$

$$\eta_2 = y + \frac{1}{\beta} \sqrt{(x - \xi)^2 - \beta^2 z^2}$$

The integral form of the velocity potential at any point  $(x,y,z)$  due to a distribution of such sources over the  $\xi\eta$ -plane is

$$\phi(x,y,z,t) = -\frac{1}{2\beta\pi} \int_0^{x-\beta z} \int_{\eta_1}^{\eta_2} W(\xi,\eta) \phi_1 \, d\eta \, d\xi \quad (8)$$

where  $W(\xi,\eta)$  represents the space variation of source strength.

The limits  $\eta_1$  and  $\eta_2$  are the intersections of the  $\xi\eta$ -plane and the upstream-opening Mach cone with vertex at point  $(x,y,z)$ . Thus the velocity potential at point  $(x,y,z)$  is affected only by the sources in the  $\xi\eta$ -plane that are within this Mach cone. (See fig. 2(a) for the limiting case  $z = 0$ .)

In order to obtain  $\phi_{Na}$  for an oscillating flat wing, the time variation of source strength  $w(t)$  is defined as follows:

$$w(t) = e^{i\omega t}$$

This form for  $w(t)$  gives for the numerator of equation (7)

$$\begin{aligned} w(t - \tau_1) + w(t - \tau_2) &= e^{i\omega(t-\tau_1)} + e^{i\omega(t-\tau_2)} \\ &= 2e^{i\omega t} e^{-i\omega \frac{\tau_2 + \tau_1}{2}} \cos \omega \frac{\tau_2 - \tau_1}{2} \end{aligned} \quad (9)$$

The space variation of source strength is defined as

$$W(\xi,\eta) = -V\alpha_0 \quad (10)$$

If the expressions given in equations (9) and (10) are substituted into equation (8) and  $z$  is made zero, the following equation for the velocity potential on the upper surface of the wing, with  $\alpha = \alpha_0 e^{i\omega t}$ , is obtained:

$$\phi_{N\alpha}(x,y,t) = \frac{V\alpha}{\beta\pi} \int_0^x \int_{\eta_1}^{\eta_2} \frac{e^{-\frac{i\omega M}{c\beta^2}(x-\xi)} \cos \frac{\omega}{\beta c} \sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} d\eta d\xi \quad (11)$$

The integrations indicated in equation (11) can be carried out in the form of a series of Bessel functions as in reference 3; however, this is not required here since, as is pointed out in the introduction, only the first-order effects of the frequency  $\omega$  are necessary in the present discussion. The terms involving higher powers of the frequency can be deleted by expanding the integrand of equation (11) into a power series in  $\omega$  and dropping the terms involving this parameter to powers higher than the first. When this expansion is made, the indicated integration in equation (11) can be carried out in closed form. There is obtained

$$\begin{aligned} \phi_{N\alpha} &\approx \frac{V\alpha}{\beta} \int_0^x \int_{\eta_1}^{\eta_2} \frac{1 - \frac{i\omega M}{c\beta^2}(x-\xi)}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} d\eta d\xi \\ &= \frac{V\alpha}{\beta} \left( x - \frac{i\omega M}{2\beta^2 c} x^2 \right) \\ &= \frac{V\alpha_0 e^{i\omega t}}{\beta} \left( x - \frac{i\omega M}{2\beta^2 c} x^2 \right) \end{aligned}$$

or, if the out-of-phase component  $i\omega\alpha$  is denoted by  $\dot{\alpha}$ ,

$$\phi_{N\alpha} \approx \frac{V\alpha}{\beta} x - \frac{V\dot{\alpha}M}{2\beta^3 c} x^2 \quad (12)$$

Note that even though the expression for  $\phi_{N\alpha}$  in equation (12) does not contain the integrated effects of higher-order terms in  $\omega$ , it remains sinusoidal with respect to time.

In order to obtain  $\phi_{N\dot{\alpha}}$  it is only necessary to change the definition of  $W(\xi, \eta)$  in equation (10). In this case

$$W(\xi, \eta) = i\omega\alpha_0(\xi - x_0) \quad (13)$$

Substituting the expressions from equations (9) and (13) into equation (8) and letting  $z$  approach zero yields

$$\phi_{N\dot{\alpha}} = \frac{i\omega\alpha}{\pi\beta} \int_0^x \int_{\eta_1}^{\eta_2} \frac{(\xi - x_0) e^{-\frac{i\omega M}{c\beta^2}(x-\xi)} \cos \frac{\omega}{\beta c} \sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} d\eta d\xi \quad (14)$$

which, after expansion to first power of  $\omega$ , yields

$$\phi_{N\dot{\alpha}} = \frac{i\omega\alpha}{\pi\beta} \int_0^x \int_{\eta_1}^{\eta_2} \frac{(\xi - x_0) \left[ 1 - \frac{i\omega M}{c\beta^2}(x - \xi) \right]}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} d\eta d\xi$$

or

$$\begin{aligned} \phi_{N\dot{\alpha}} &= \frac{i\omega\alpha}{2\beta} x(x - 2x_0) \\ &= \frac{\dot{\alpha}}{2\beta} (x^2 - 2xx_0) \end{aligned} \quad (15)$$

Equations (12) and (15) may be substituted into equation (5) to obtain the complete expression for  $\phi_N$  at the upper surface of the wing.

Derivation of  $\phi_T$ .—As pointed out in the introduction, distributions of doublets may be used to satisfy the boundary-value problem in the region of mixed supersonic flow (region T). The type of doublet required is that with axis normal to the reference plane. It will be recalled that the potential of such a doublet may be obtained from the potential of a source, located in the reference plane, by a partial differentiation of the source potential with respect to the direction normal to the reference plane.

If such a differentiation is applied to the potential of a moving source in the  $\xi\eta$ -plane (equation (7)), the following expression that may be shown to satisfy equation (1) is obtained:

$$\phi_2 = \frac{\partial}{\partial z} \left[ W(\xi, \eta) \frac{w(t - \tau_1) + w(t - \tau_2)}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} \right] \quad (16)$$

where now  $W(\xi, \eta)$  refers to doublet strength.

In order to obtain  $\phi_{T\alpha}$  for the oscillating flat wing it is convenient to assign to the functions  $w(t - \tau_1) + w(t - \tau_2)$  and  $W(\xi, \eta)$  the expressions given in equation (9) and equation (10), respectively. Substituting these expressions into equation (16) gives

$$\begin{aligned} \phi_2 &= \frac{\partial}{\partial z} \left[ -2V\alpha \frac{\frac{-i\omega M}{e c \beta^2}(x - \xi) \cos \frac{\omega}{\beta c} \sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} \right] \\ &= -2V\alpha \beta^2 z e \frac{-i\omega M}{c \beta^2}(x - \xi) \left\{ \frac{\omega \sin \frac{\omega}{\beta c} \sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\beta^2 c (\eta - \eta_1)(\eta_2 - \eta)} \right. \\ &\quad \left. + \frac{\cos \frac{\omega}{\beta c} \sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{[(\eta - \eta_1)(\eta_2 - \eta)]^{3/2}} \right\} \quad (17) \end{aligned}$$

Expanding the terms involving  $\omega$  in equation (17) into a power series to the first order of  $\omega$  gives

$$\begin{aligned} \phi_2 &\approx -2V\alpha\beta^2z \frac{1 - \frac{i\omega M}{c\beta^2}(x - \xi)}{\left[(\eta - \eta_1)(\eta_2 - \eta)\right]^{3/2}} \\ &= -2V\beta^2z \left\{ \left( \alpha - \frac{M\ddot{\alpha}x}{c\beta^2} \right) \frac{1}{\left[(\eta - \eta_1)(\eta_2 - \eta)\right]^{3/2}} + \frac{M\dot{\alpha}}{c\beta^2} \frac{\xi}{\left[(\eta - \eta_1)(\eta_2 - \eta)\right]^{3/2}} \right\} \quad (18) \end{aligned}$$

Each of the two terms within the braces in equation (18) can be shown to satisfy equation (1) to the first order of  $\omega$ . Furthermore it can be seen that, to the first order of  $\omega$ , the potential of a moving doublet may be considered as a superposition of two doublets of the stationary type; one of strength proportional to  $\alpha - \frac{M\ddot{\alpha}x}{c\beta^2}$  and the other of strength proportional to  $\frac{M\dot{\alpha}}{c\beta^2}\xi$ .

The potential  $\phi_{T\alpha}$  composed of distributions of doublets of the types in equation (18) may be written

$$\begin{aligned} \phi_{T\alpha} &= -\frac{V\beta z}{\pi} \left( \alpha - \frac{M\ddot{\alpha}x}{c\beta^2} \right) \iint_S \frac{F_1(\xi, \eta) d\eta d\xi}{\left[(\eta - \eta_1)(\eta_2 - \eta)\right]^{3/2}} \\ &\quad - \frac{VzM\dot{\alpha}}{c\beta\pi} \iint_S \frac{\xi F_2(\xi, \eta) d\eta d\xi}{\left[(\eta - \eta_1)(\eta_2 - \eta)\right]^{3/2}} \quad (19) \end{aligned}$$

where the integration is to be extended over the wing surface  $S$  included in the fore cone with vertex at  $(x, y, z)$  (see fig. 2(b) for the limiting case  $z = 0$ ) and where  $F_1$  and  $F_2$  are functions describing the manner in which the doublets are to be distributed in the region  $S$  and must be determined in such a manner that the boundary conditions (a) and (b) will be satisfied.

The normal procedure of determining  $F_1$  and  $F_2$  would be to formulate an integral equation by imposing the boundary conditions on equation (19); however, this procedure is tedious and difficult. It can be circumvented here by making use of known solutions to similar problems. The integrand of the first integral in equation (19) is recognized to be the same as that in the integral form of the velocity potential for a similar wing at constant angle of attack, and the integrand of the second integral is recognized to be the same as that in the integral form of the velocity potential for a similar wing undergoing a constant rotation about its leading edge. Furthermore, the boundary conditions for these cases can be stated precisely as conditions (a) and (b), stated herein, for the oscillating wing. The distribution function and the velocity potential at the wing surface for the rectangular wing at constant angle of attack can be readily obtained from the known distribution of lift for this case. The distribution function and velocity potential at the wing surface for the rectangular wing undergoing a constant rotation about its leading edge can be obtained from the wing at constant angle of attack by linear superposition. The distribution functions and velocity potentials at the wing surface for both these cases are derived in the appendix. The value of  $\phi_{T\alpha}$  (equation 19) at the wing surface can be easily deduced from these results; that is,

$$\phi_{T\alpha}(x, y, t) \approx \frac{2V}{\beta\pi} \left( \alpha - \frac{M\dot{\alpha}x}{c\beta^2} \right) \left[ \sqrt{\beta y(x - \beta y)} + x \sin^{-1} \sqrt{\frac{\beta y}{x}} \right] \\ + \frac{VM\dot{\alpha}}{c\beta^3\pi} \left[ \frac{5x - 2\beta y}{3} \sqrt{\beta y(x - \beta y)} + x^2 \sin^{-1} \sqrt{\frac{\beta y}{x}} \right]$$

$$\begin{aligned}
&= \frac{2V\alpha}{\beta\pi} \left[ \sqrt{\beta y(x - \beta y)} + x \sin^{-1} \sqrt{\frac{\beta y}{x}} \right] \\
&- \frac{M^2 \dot{\alpha}}{3\pi\beta^3} \left[ (x + 2\beta y) \sqrt{\beta y(x - \beta y)} + 3x^2 \sin^{-1} \sqrt{\frac{\beta y}{x}} \right] \quad (20)
\end{aligned}$$

In order to obtain  $\phi_{T\dot{\alpha}}$  it is convenient to define  $W(\xi, \eta)$  as in equation (13). Substituting equations (9) and (13) into equation (16) yields

$$\phi_2 = \frac{\partial}{\partial z} \left[ -2\dot{\alpha}(\xi - x_0) \frac{e^{-\frac{1}{c\beta^2}(\xi - \eta)} \cos \frac{\omega}{\beta c} \sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} \right] \quad (21)$$

Carrying out the indicated differentiation and expanding to the first power of  $\omega$  results in

$$\begin{aligned}
\phi_2 &= -2\beta^2 z \dot{\alpha} \frac{\xi - x_0}{\left[ (\eta - \eta_1)(\eta_2 - \eta) \right]^{3/2}} \\
&= 2\beta^2 z \dot{\alpha} \left\{ \frac{x_0}{\left[ (\eta - \eta_1)(\eta_2 - \eta) \right]^{3/2}} - \frac{\xi}{\left[ (\eta - \eta_1)(\eta_2 - \eta) \right]^{3/2}} \right\} \quad (22)
\end{aligned}$$

The two expressions in the right side of equation (22) have the same form as the expressions in the right side of equation (18). The following

integral expression for  $\phi_{T\dot{\alpha}}$  is therefore of the same form as the integral expression for  $\phi_{T\alpha}$  (equation 19) and the functions  $F_3$  and  $F_4$  can be determined by the procedure given for determining  $F_1$  and  $F_2$ :

$$\phi_{T\dot{\alpha}} = \frac{\beta z \dot{\alpha} x_0}{\pi} \iint_S \frac{F_3(\xi, \eta) d\xi d\eta}{[(\eta - \eta_1)(\eta_2 - \eta)]^{3/2}} - \frac{\beta z \dot{\alpha}}{\pi} \iint_S \frac{\xi F_4(\xi, \eta) d\xi d\eta}{[(\eta - \eta_1)(\eta_2 - \eta)]^{3/2}}$$

The expression for  $\phi_{T\dot{\alpha}}$  at the wing surface can therefore be deduced from the results in the appendix to be

$$\begin{aligned} \phi_{T\dot{\alpha}} &= -\frac{2\dot{\alpha}x_0}{\beta\pi} \left[ \sqrt{\beta y(x - \beta y)} + x \sin^{-1} \sqrt{\frac{\beta y}{x}} \right] \\ &\quad + \frac{2\dot{\alpha}}{\beta\pi} \left[ \frac{5x - 2\beta y}{6} \sqrt{\beta y(x - \beta y)} + \frac{x^2}{2} \sin^{-1} \sqrt{\frac{\beta y}{x}} \right] \\ &= \frac{\dot{\alpha}}{\beta\pi} \left[ \frac{5x - 2\beta y - 6x_0}{3} \sqrt{\beta y(x - \beta y)} + x(x - 2x_0) \sin^{-1} \sqrt{\frac{\beta y}{x}} \right] \end{aligned} \quad (23)$$

Equations (20) and (23) may be substituted into equation (6) to obtain  $\phi_T$  at the wing surface.

Aerodynamic moment.— The local pressure difference between the upper and lower surfaces may be written as

$$p = -2\rho \left( \frac{\partial \phi}{\partial t} + V \frac{\partial \phi}{\partial x} \right) \quad (24)$$

The aerodynamic moment on the entire wing about the axis  $x = x_0$ , measured in the positive clockwise direction, is then (see fig. 1)

$$M_\alpha = 2 \iint_N (x - x_0) p_N dy dx + 2 \iint_T (x - x_0) p_T dy dx \quad (25)$$

If all linear dimensions involved in equations (24) and (25) are referred to the chord  $2b$ , these equations become, in a sense, nondimensional and can be written

$$p = -2\rho \left( \frac{\partial \phi}{\partial t} + \frac{V}{2b} \frac{\partial \phi}{\partial x} \right) \quad (26)$$

and

$$M_\alpha = 16b^3 \iint_N (x - x_0) p_N dy dx + 16b^3 \iint_T (x - x_0) p_T dy dx \quad (27)$$

Putting in the appropriate limits of integration (see fig. 2) results in the following equation for the total aerodynamic moment:

$$M_\alpha = 16b^3 \int_0^1 \int_{x/\beta}^h (x - x_0) p_N dy dx + 16b^3 \int_0^1 \int_0^{x/\beta} (x - x_0) p_T dy dx \quad (28)$$

If the nondimensional forms of  $p_N$  and  $p_T$  are computed from equations (12), (15), (20), and (23) and substituted into equation (28), there is obtained a complex expression for  $M_\alpha$  that may be written as

$$M_\alpha = -8\rho b^3 V^2 k^2 \alpha A (M_3 + iM_4) \quad (29)$$

where

$$M_3 = \frac{1}{3\beta^2 k^2} \left[ 3\beta(1 - 2x_0) - \frac{1}{A}(2 - 3x_0) \right] \quad (30)$$

is the in-phase component of the aerodynamic-moment coefficient, and

$$M_4 = \frac{2}{3\beta^4 k} \left[ 2\beta^3(1 - 3x_0 + 3x_0^2) - \beta(2 - 3x_0) - \frac{2\beta^2 x_0(3x_0 - 2) + (4x_0 - 3)}{2A} \right] \quad (31)$$

is the out-of-phase or torsional-damping aerodynamic-moment coefficient.

The expression for  $M_4$  in equation (31) represents the central result of the present investigation because the conditions of torsional stability or instability depend directly on the sign and magnitude of  $M_4$ . It will be noted, in equation (31), that  $M_4$  may undergo changes in sign as any one of the parameters  $\beta$ ,  $x_0$ , and  $A$  is varied.

Although equation (31) has been derived for the condition that the intersection of the Mach cones from the foremost points of the side edges is off the wing, it can be shown that it remains valid as long as the Mach cone from one side edge does not intersect the opposite side edge. When a side edge is intersected by the Mach cone from the opposite edge different velocity potentials from those explicitly derived herein are required for portions of the wing behind the points where the Mach cone intersects the edge. The derivation of these potentials is not considered here.

Total torsional-damping coefficient and some selected calculations of stability conditions.— If mechanical damping is assumed to exist about the axis of rotation, for example as the hinge friction in an aileron installation, and if this damping is converted to coefficient form as in references 3 and 9 and combined with the aerodynamic damping, the total torsional-damping coefficient may be written as

$$M_4^* = M_4(\beta, x_0, k, A) + \mu r_\alpha 2 \left( \frac{\omega_\alpha}{\omega} \right)^2 g_\alpha$$

or

$$M_4' = M_4(\beta, x_0, k, A) + \frac{1}{k^2} \mu r_\alpha^2 \left( \frac{b a_\alpha}{V} \right)^2 g_\alpha \quad (32)$$

Negative values of  $M_4'$  indicate dynamic instability that may correspond to single-degree torsional flutter. Positive values of  $M_4'$  indicate stable conditions and between the stable and unstable conditions, that is when  $M_4'$  vanishes, a borderline state of unstable equilibrium separating damped and undamped torsional oscillations occurs. Since the term involving  $g_\alpha$  in equation (32) is always positive, negative values of  $M_4'$  and the vanishing of  $M_4'$  are associated with negative values only of  $M_4$  (equation (31)).

With the exception of aspect ratio  $A$ , the effects of the individual parameters in equation (32) are presented in references 3 and 4. The effect of aspect ratio for some particular values of the other parameters is briefly discussed in the following paragraphs.

The range of values of  $x_0$  and  $\beta^2$  (and  $M^2$ ) for which  $M_4'$  vanishes when  $g_\alpha$  is assumed to be zero and  $A$  has different selected values is shown plotted in figure 3. The regions inside the different curves give the range of values of  $x_0$ ,  $\beta^2$ , and  $M^2$  in which instability might occur. The curve for infinite aspect ratio agrees with that given in figure 21 of reference 3 and that given in figure 2 of reference 4. It will be noted that decreasing the aspect ratio has a highly stabilizing effect.

The dashed portions of the curves in figure 3 correspond to aspect ratio and Mach number combinations that cause the Mach cone from the foremost point of one tip to intersect the opposite tip ahead of the trailing edge.

In figure 4 the quantity  $M_4$  multiplied, for convenience, by  $3\beta^4 k$  is plotted as a function of  $M^2$  for the values of  $A$  selected in figure 3. The ordinates of these curves are proportional to the amount of negative aerodynamic damping available for given Mach numbers and for given values of the frequency parameter  $k$ . The stabilizing effect of decreasing aspect ratio is apparent in this figure.

Corresponding values of the flutter-speed coefficient  $V/b a_\alpha$  and aspect ratio  $A$  for which  $M_4'$  vanishes for some selected values of the parameters  $x_0$ ,  $g_\alpha$ ,  $M^2$ ,  $\mu$ ,  $\frac{1}{k}$ , and  $r_\alpha^2$  are plotted in figure 5.

The regions above these curves give the range of values in which instabilities might occur.

It should be recalled that thickness effects have not been considered in the present discussion. Second-order thickness effects may be of considerable significance in regard to undamped torsional oscillations of wings of finite or infinite aspect ratio.

Langley Aeronautical Laboratory  
National Advisory Committee for Aeronautics  
Langley Air Force Base, Va., March 28, 1949

## APPENDIX

VELOCITY POTENTIALS FOR TIP REGION OF RECTANGULAR  
WING IN STEADY SUPERSONIC FLOW

It can be shown, as in reference 10, that at points in the  $xy$ -plane the perturbation-velocity potential in a supersonic stream due to doublets in the  $xy$ -plane distributed according to a distribution function  $f(\xi, \eta)$  is proportional to the distribution function; that is,

$$\begin{aligned}\phi(x, y, 0) &= \lim_{z \rightarrow 0} \frac{V\alpha\beta z}{\pi} \iint_S \frac{f(\xi, \eta) d\eta d\xi}{[(x - \xi)^2 - \beta^2(y - \eta)^2 - \beta^2 z^2]^{3/2}} \\ &= \pm \pi f(x, y)\end{aligned}\tag{A1}$$

where the positive sign applies to the upper surface of the region  $S$  and the negative sign applies to the lower surface; that is, the direction from which  $z \rightarrow 0$ . When the appropriate distribution function can be found, such doublet distributions can be used to satisfy boundary-value problems, in the  $xy$ -plane, for vanishingly thin airfoils at vanishingly small angles of attack. In the case of a rectangular wing placed in a supersonic stream flowing from the negative  $x$ -direction, with its leading edge normal to the free-stream direction, the added velocity in the free-stream direction in the tip region can be obtained from the expression derived for the added pressure difference, in this region, in reference 11. In the present notation this expression for pressure difference is

$$2\rho V \frac{\partial \phi}{\partial x} = \frac{4\alpha\rho V^2}{\beta\pi} \sin^{-1} \sqrt{\frac{\beta y}{x}}$$

or

$$\frac{\partial \phi}{\partial x} = 2 \frac{V\alpha}{\beta\pi} \sin^{-1} \sqrt{\frac{\beta y}{x}}\tag{A2}$$

If  $\phi$  is eliminated from equations (A1) and (A2), the following differential equation for  $f(x,y)$  is obtained:

$$\pi \frac{\partial f}{\partial x} = \frac{2V\alpha}{\beta\pi} \sin^{-1} \sqrt{\frac{\beta y}{x}} \quad (A3)$$

The expression for  $\pi f(x,y)$  or  $\phi(x,y)$  can now be obtained by a partial integration, with respect to  $x$ , of equation (A3). This integration gives

$$\pi f(x,y) = \phi(x,y) = \frac{2V\alpha}{\beta\pi} \left[ \sqrt{\beta y(x - \beta y)} + x \sin^{-1} \sqrt{\frac{\beta y}{x}} \right] \quad (A4)$$

For the rectangular wing rotating about its leading edge the distribution function for the tip region can be obtained by a linear superposition of the function given in (A4); that is,

$$\begin{aligned} -\pi \int_0^x f(x - \xi) d\xi &= -\frac{2V\alpha}{\beta\pi} \int_0^x \left\{ \sqrt{\beta y [(x - \xi) - \beta y]} + (x - \xi) \sin^{-1} \sqrt{\frac{\beta y}{x - \xi}} \right\} d\xi \\ &= \frac{V\alpha}{\beta\pi} \left[ \frac{5x - 2\beta y}{3} \sqrt{\beta y(x - \beta y)} + x^2 \sin^{-1} \sqrt{\frac{\beta y}{x}} \right] \quad (A5) \end{aligned}$$

If in equation (A4)  $\alpha$  is replaced by  $\alpha - \frac{M\dot{\alpha}x}{c\beta^2}$  and if in equation (A5)

$\alpha$  is replaced by  $\frac{M\dot{\alpha}}{c\beta^2}$ , the sum of the resulting equations will yield

equation (20) of the text. Similarly, if in equation (A4)  $V\alpha$  is replaced by  $-x_0\dot{\alpha}$  and if in equation (A5)  $V\alpha$  is replaced by  $\dot{\alpha}$ , the sum of the resulting equations will yield equation (23) of the text.

## REFERENCES

1. Possio, C.: L'Azione aerodinamica sul profilo oscillante alle velocita ultrasonore. Acta, Pont. Acad. Sci., vol. I, no. 11, 1937, pp. 93-106.
2. Temple, G., and Jahn, H. A.: Flutter at Supersonic Speeds. Part I. Mid-Chord Derivative Coefficients for a Thin Aerofoil at Zero Incidence. R. & M.No. 2140, British A.R.C., 1945.
3. Garrick, I. E., and Rubinow, S. I.: Flutter and Oscillating Air-Force Calculations for an Airfoil in a Two-Dimensional Supersonic Flow. NACA Rep. 846, 1946.
4. Jones, W. P.: Negative Torsional Aerodynamic Damping at Supersonic Speeds. R. & M.No. 2194, British A.R.C., 1946.
5. Cheilek, H., and Frissel, H.: Theoretical Criteria for Single Degree of Freedom Flutter at Supersonic Speeds. Rep. No. CAL-7A, Cornell Aero. Lab., May 8, 1947.
6. Garrick, I. E., and Rubinow, S. I.: Theoretical Study of Air Forces on an Oscillating or Steady Thin Wing in a Supersonic Main Stream. NACA Rep. 872, 1947.
7. Evvard, John C.: A Linearized Solution for Time-Dependent Velocity Potentials near Three-Dimensional Wings at Supersonic Speeds. NACA TN 1699, 1948.
8. Harmon, Sidney M.: Stability Derivatives of Thin Rectangular Wings at Supersonic Speeds. NACA TN 1706, 1948.
9. Theodorsen, Theodore, and Garrick, I. E.: Flutter Calculations in Three Degrees of Freedom. NACA Rep. 741, 1942.
10. Brown, Clinton E., and Adams, Mac C.: Damping in Pitch and Roll of Triangular Wings at Supersonic Speeds. NACA TN 1566, 1948.
11. Lighthill, M. J.: The Supersonic Theory of Wings of Finite Span. R. & M.No. 2001, British A.R.C., 1944.

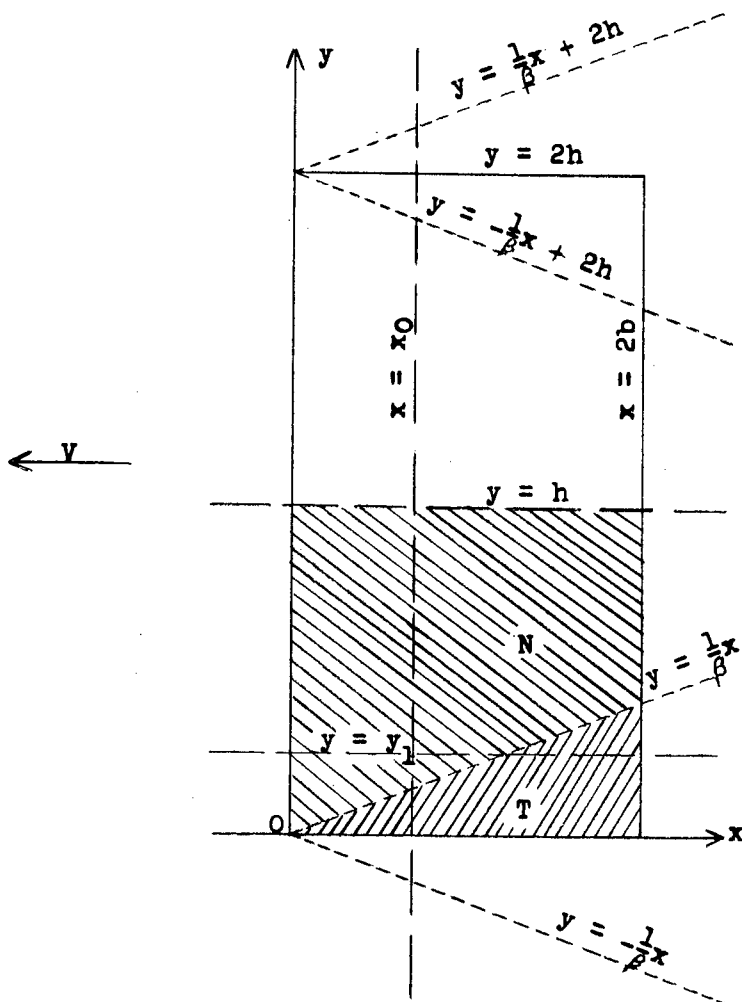
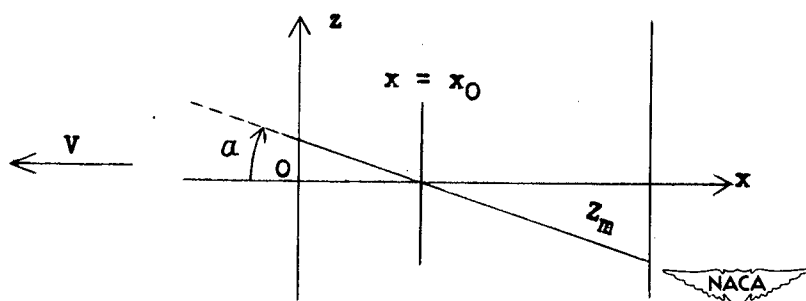
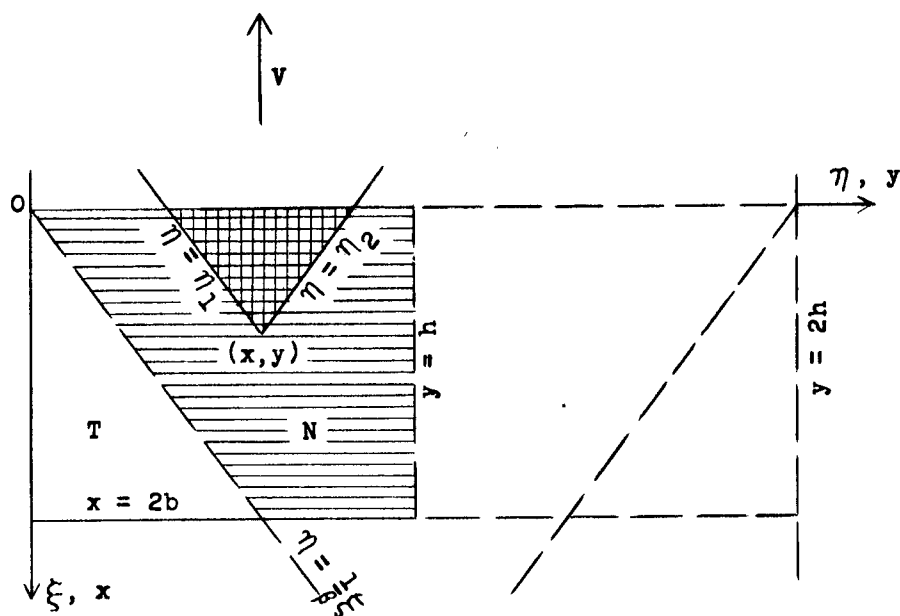
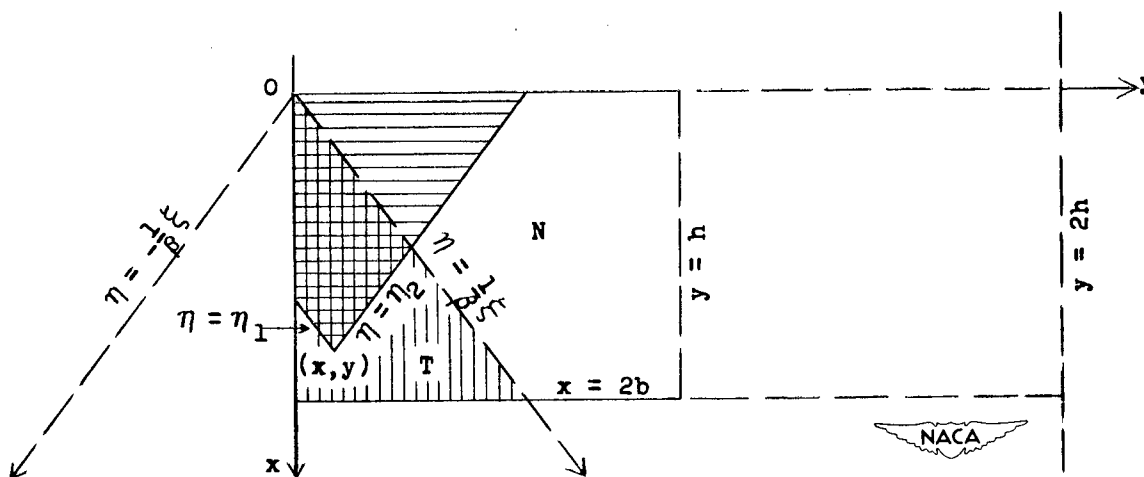
(a) Plan form ( $xy$ -plane).(b) Section  $y = y_1$  ( $xz$ -plane).

Figure 1.- Sketch illustrating chosen coordinate system of airfoil oscillating in torsion.



(a) Purely supersonic region.



(b) Mixed supersonic region.

Figure 2.- Sketch illustrating areas of integration for "purely supersonic" and "mixed supersonic" regions of flow.

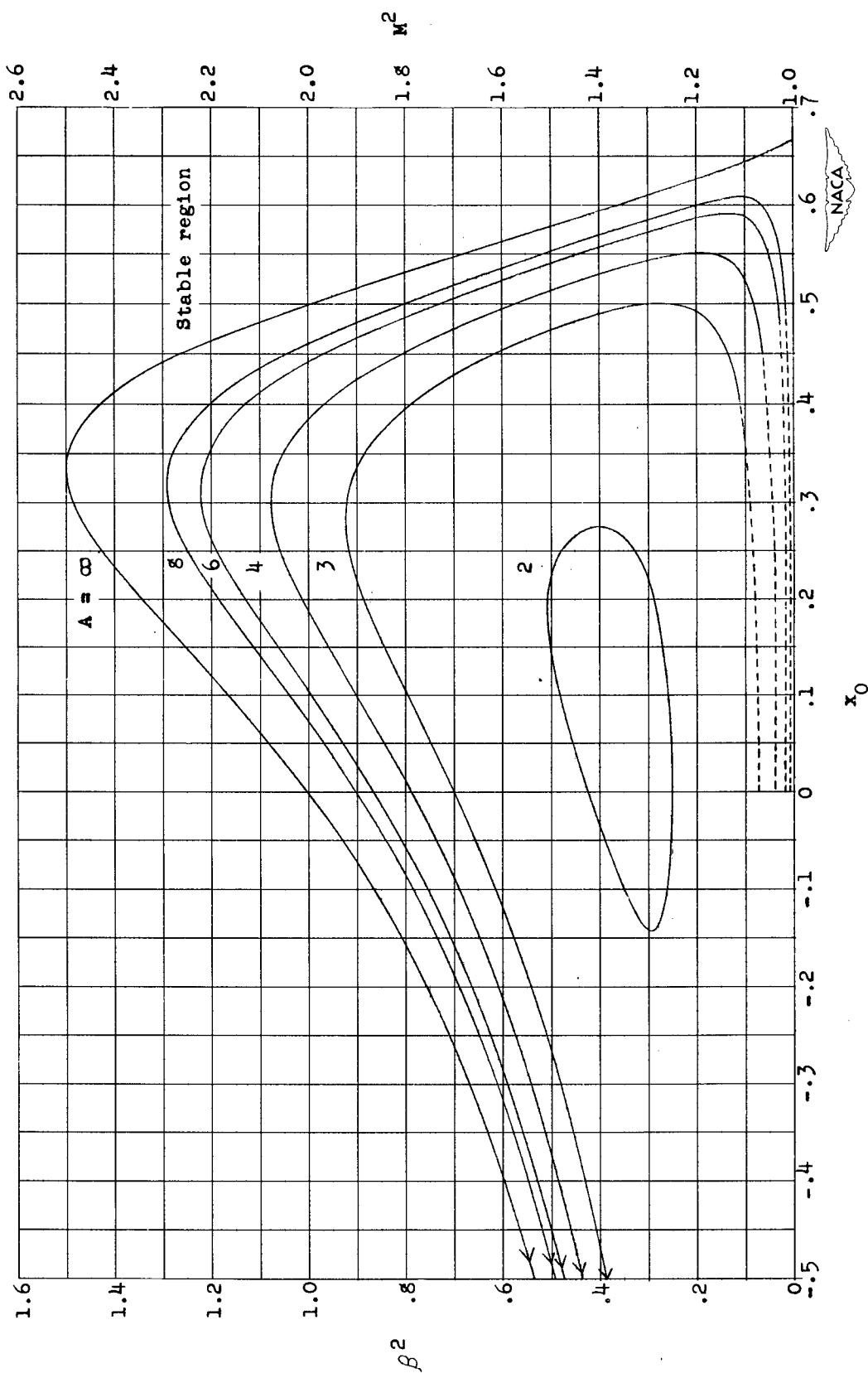
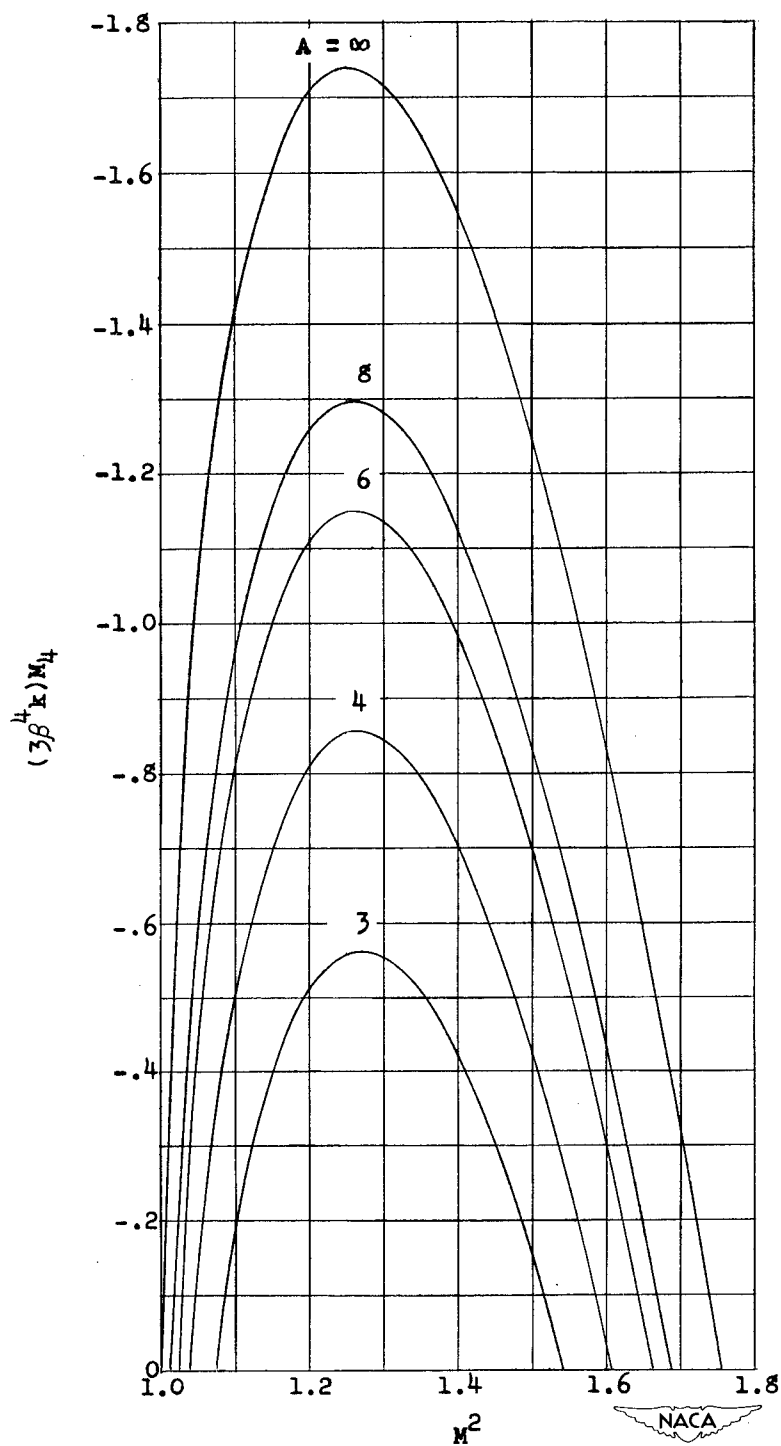
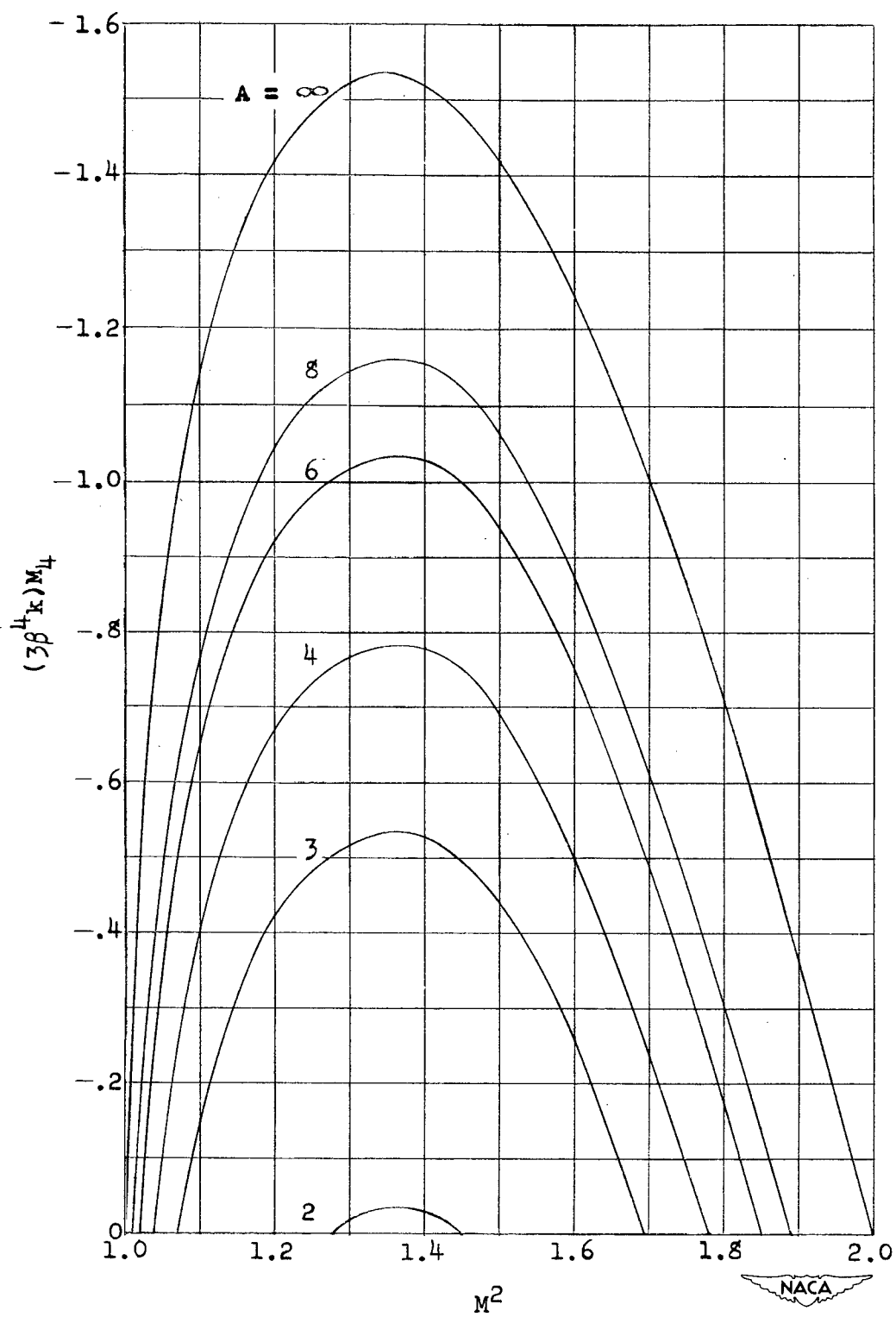


Figure 3.- Curves showing ranges of  $\beta^2$ ,  $M^2$ , and  $x_0$  for which the aerodynamic torsional damping vanishes for some selected values of  $A$ .  $g_a = 0$ .



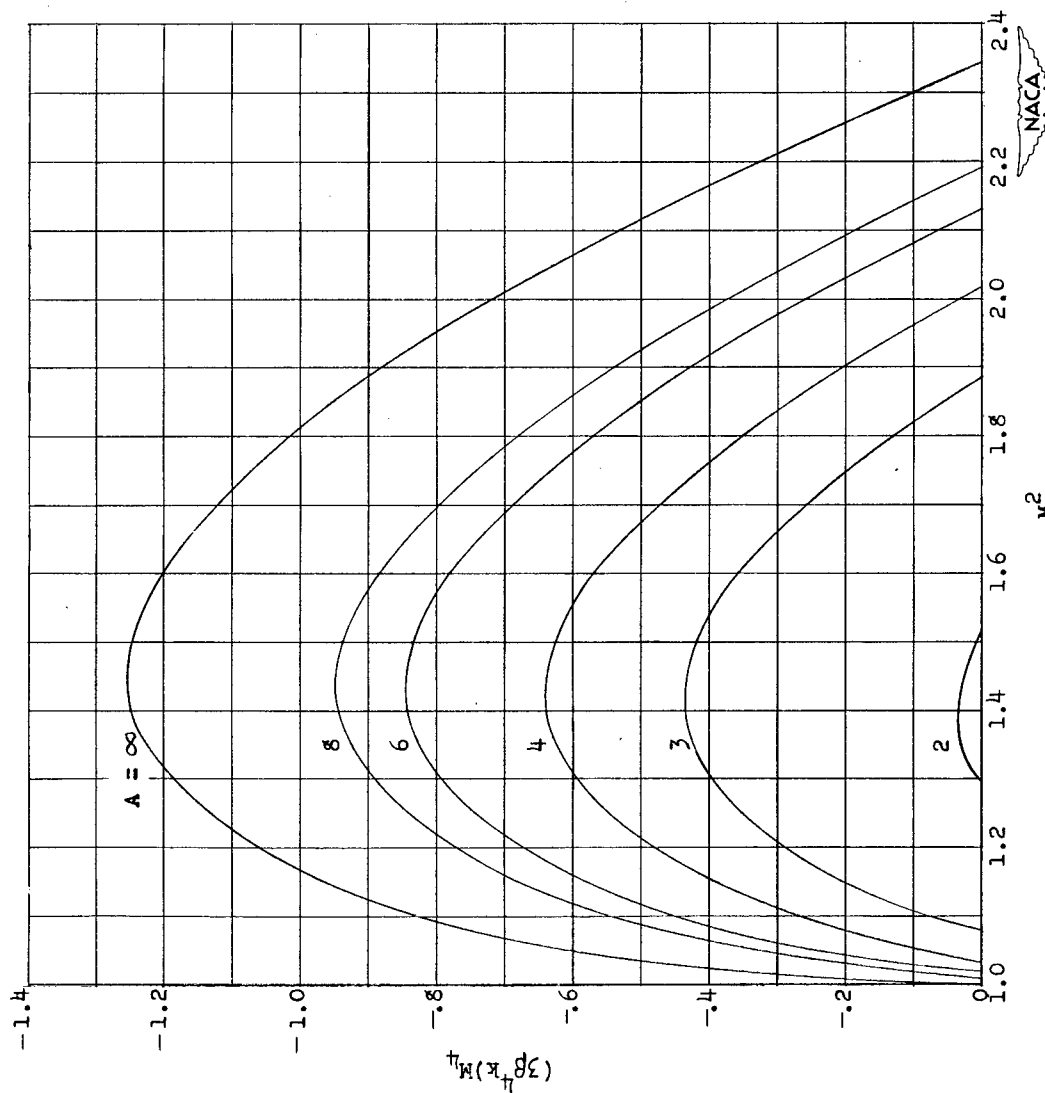
(a)  $x_0 = -0.2$ .

Figure 4.- Curves showing the variation of negative aerodynamic torsional damping with  $M^2$  for selected values of  $A$  and  $x_0$ .



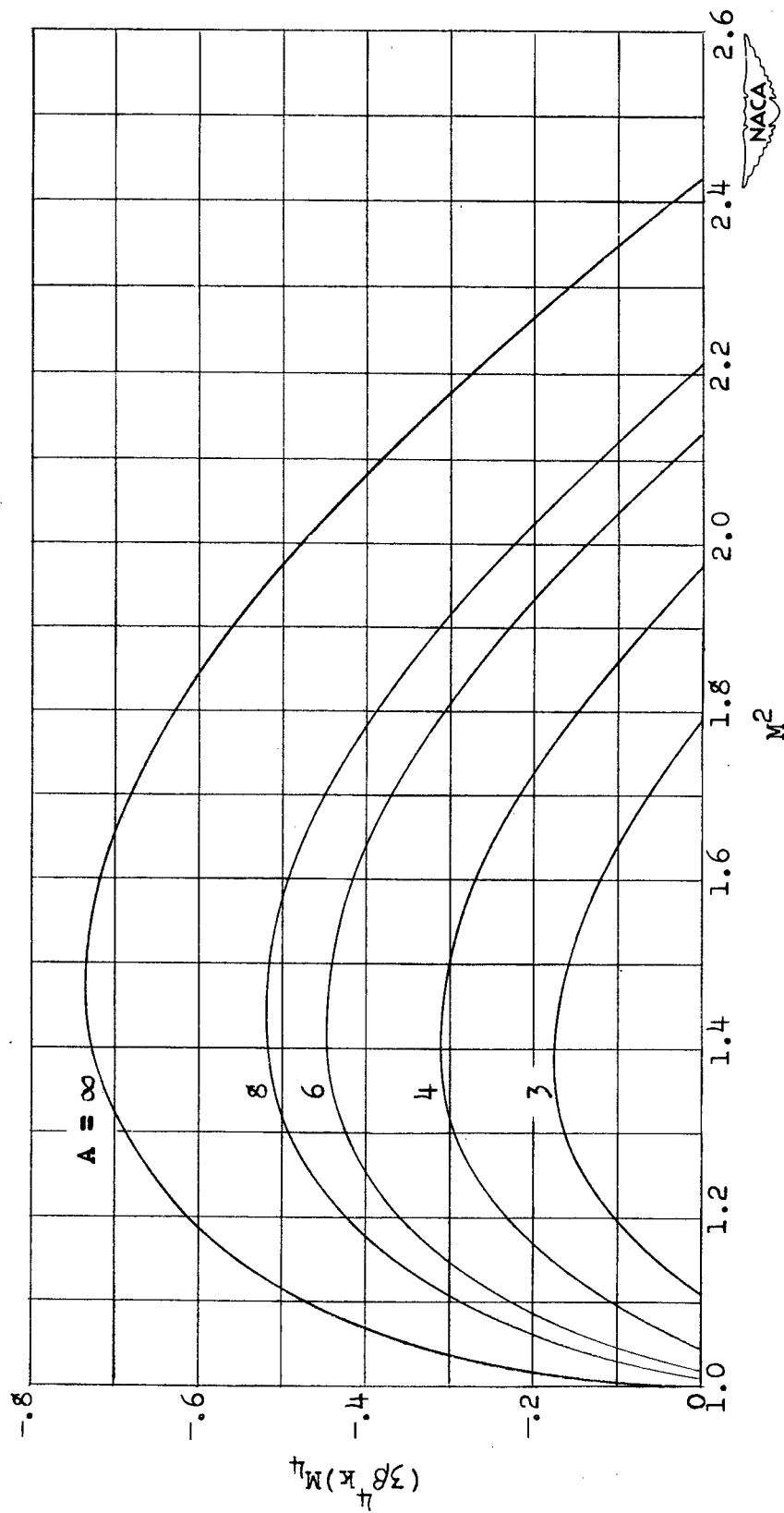
(b)  $x_0 = 0$ .

Figure 4.- Continued.



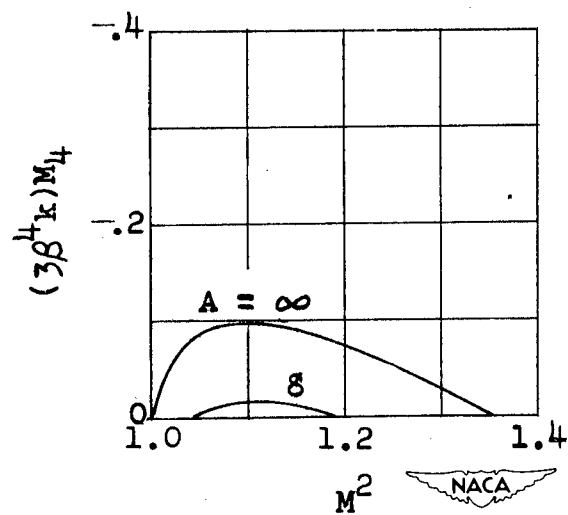
(c)  $x_0 = 0.2$ .

Figure 4.- Continued.



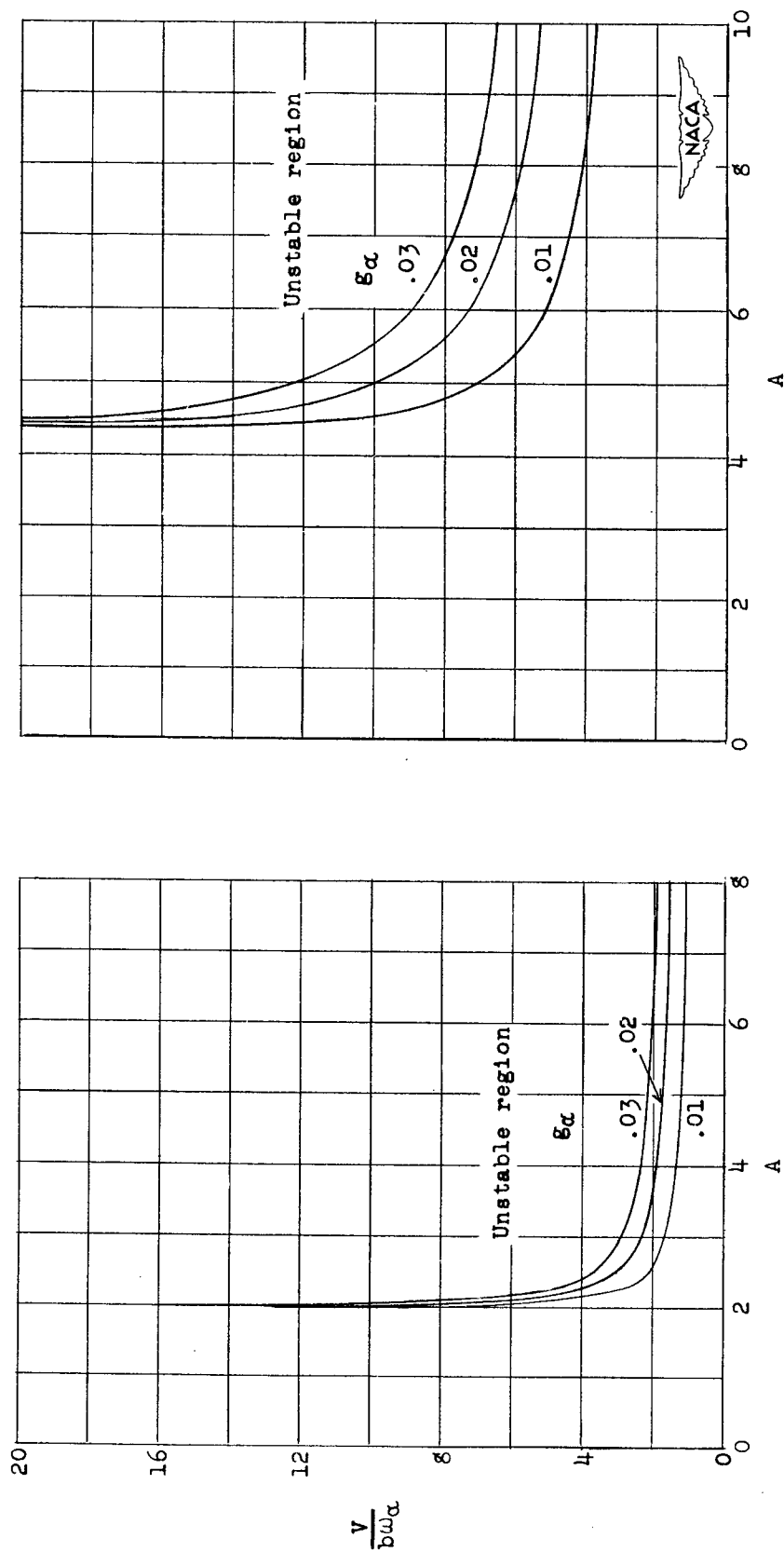
(d)  $x_0 = 0.4$ .

Figure 4.- Continued.



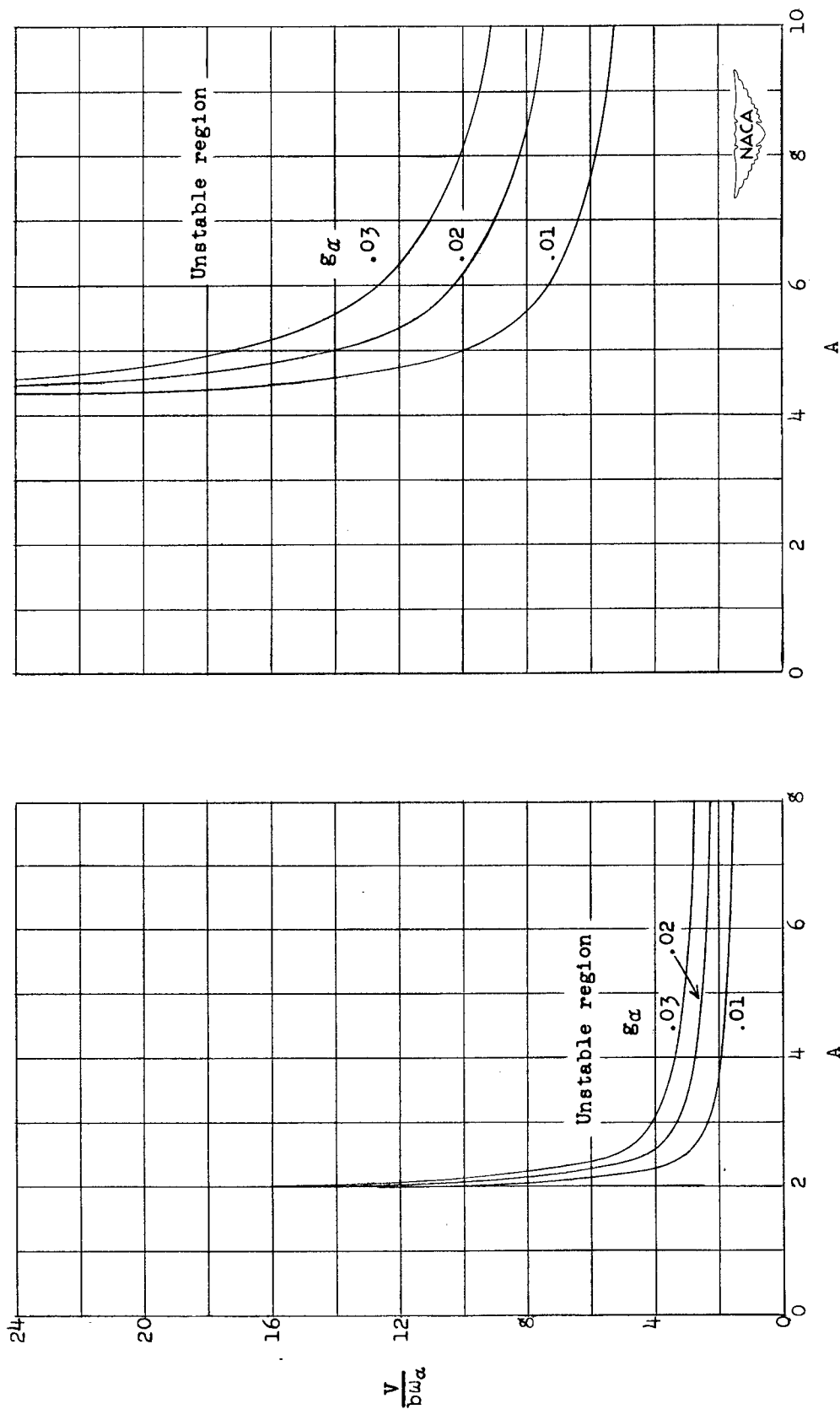
(e)  $x_0 \approx 0.6$ .

Figure 4.- Concluded.



- (a)  $M^2 = 1.4$ ;  $x_0 = 0$ ;  $\frac{1}{k} = 10$ ;  $\mu = 100$ ;  $r_a^2 = 0.3$ .  
 (b)  $M^2 = 1.8$ ;  $x_0 = 0$ ;  $\frac{1}{k} = 10$ ;  $\mu = 100$ ;  $r_a^2 = 0.3$ .

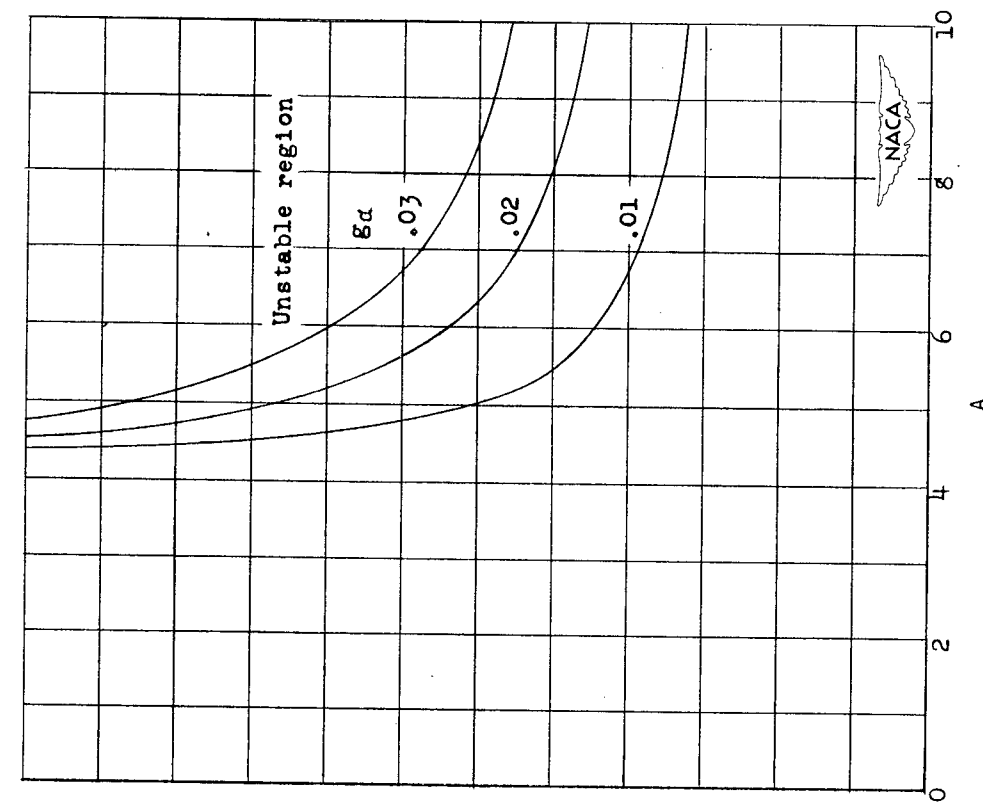
Figure 5.- Curves showing ranges of flutter coefficient and aspect ratio for which the total torsional damping coefficient vanishes for some selected values of  $M^2$ ,  $x_0$ ,  $\frac{1}{k}$ ,  $\mu$ ,  $r_a^2$ , and  $g_\alpha$ .



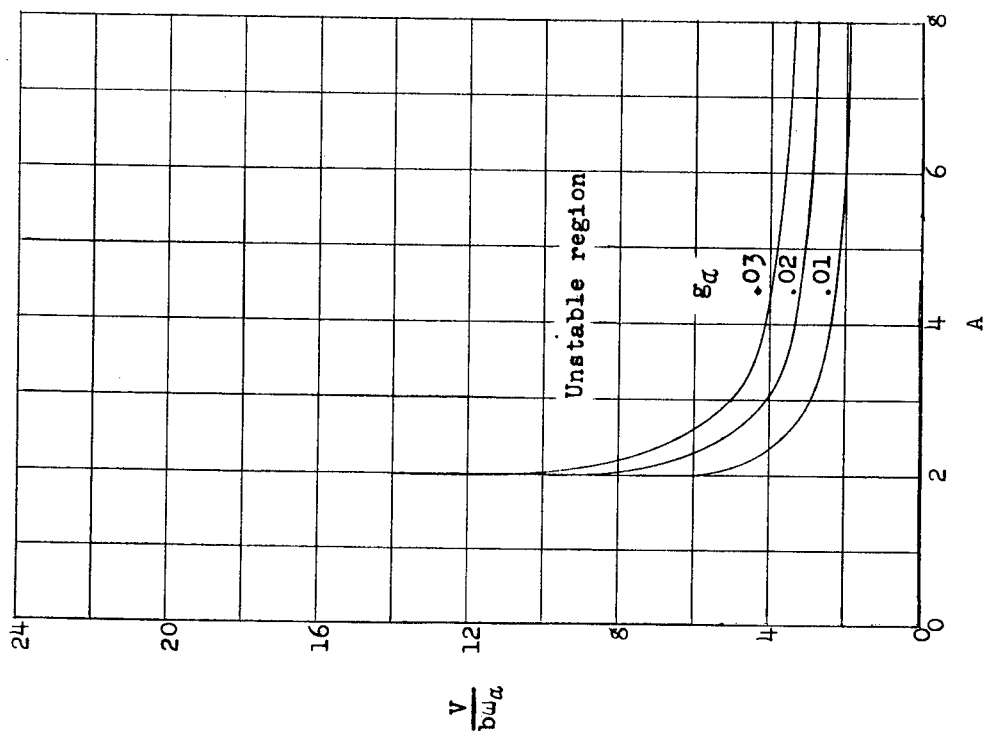
(c)  $M^2 = 1.4$ ;  $x_0 = 0$ ;  $\frac{1}{K} = 10$ ;  $\mu = 200$ ;  $r_a^2 = 0.3$ .

(d)  $M^2 = 1.8$ ;  $x_0 = 0$ ;  $\frac{1}{K} = 10$ ;  $\mu = 200$ ;  $r_a^2 = 0.3$ .

Figure 5.- Continued.

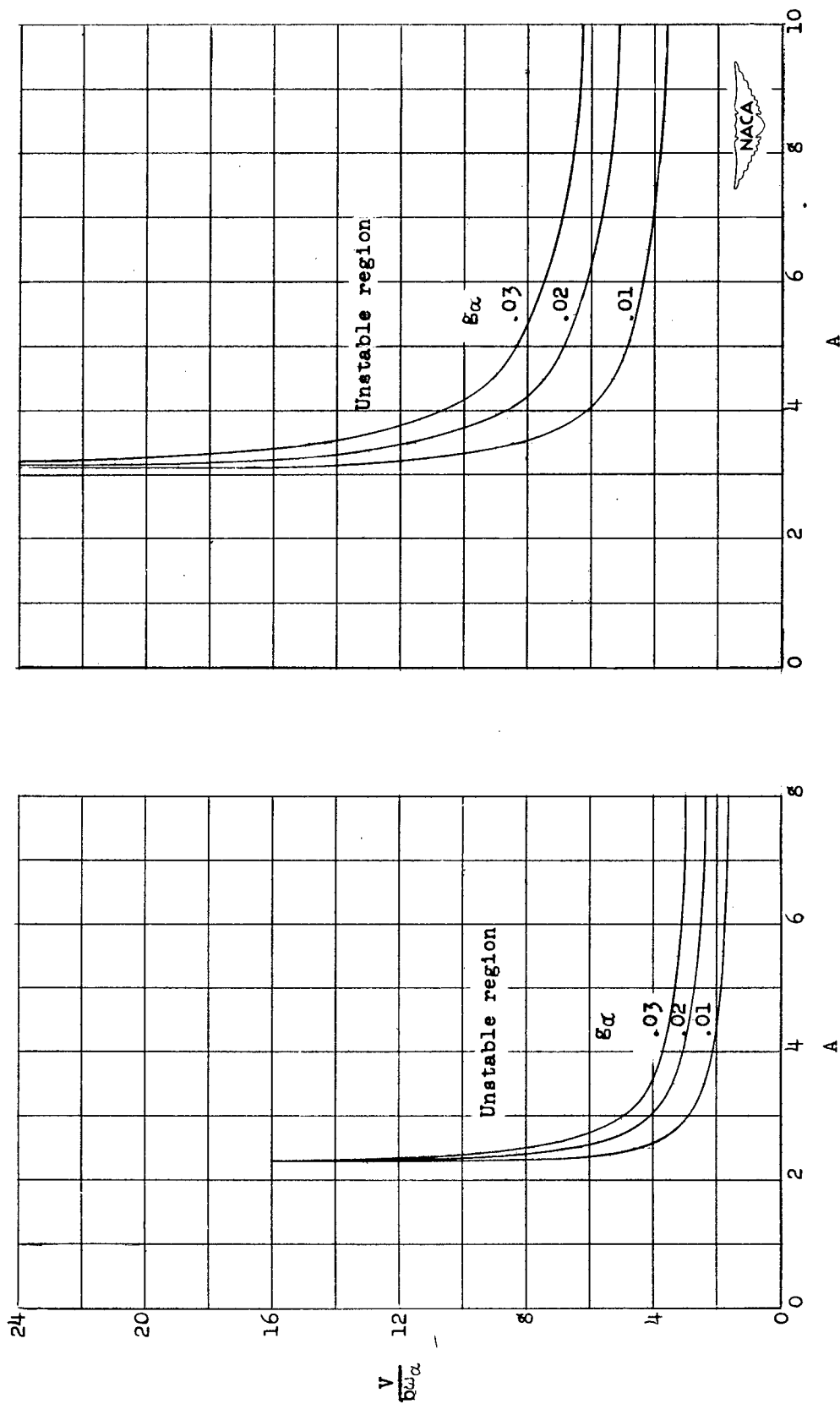


(e)  $M^2 = 1.4$ ;  $x_0 = 0$ ;  $\frac{1}{k} = 30$ ;  $\mu = 100$ ;  $r_a^2 = 0.3$ .



(f)  $M^2 = 1.8$ ;  $x_0 = 0$ ;  $\frac{1}{k} = 30$ ;  $\mu = 100$ ;  $r_a^2 = 0.3$ .

Figure 5.- Continued.



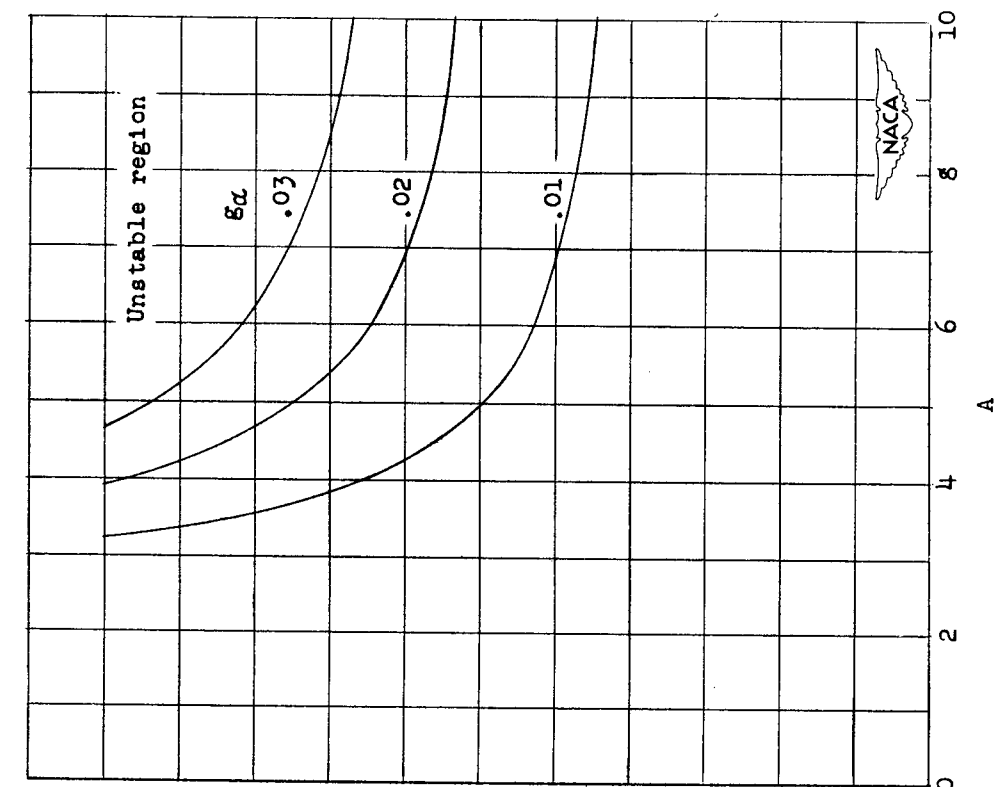
(g)  $M^2 = 1.4$ ;  $x_0 = 0.4$ ;  $\frac{1}{k} = 10$ ;  $\mu = 100$ ;

(h)  $M^2 = 1.8$ ;  $x_0 = 0.4$ ;  $\frac{1}{k} = 10$ ;  $\mu = 100$ ;

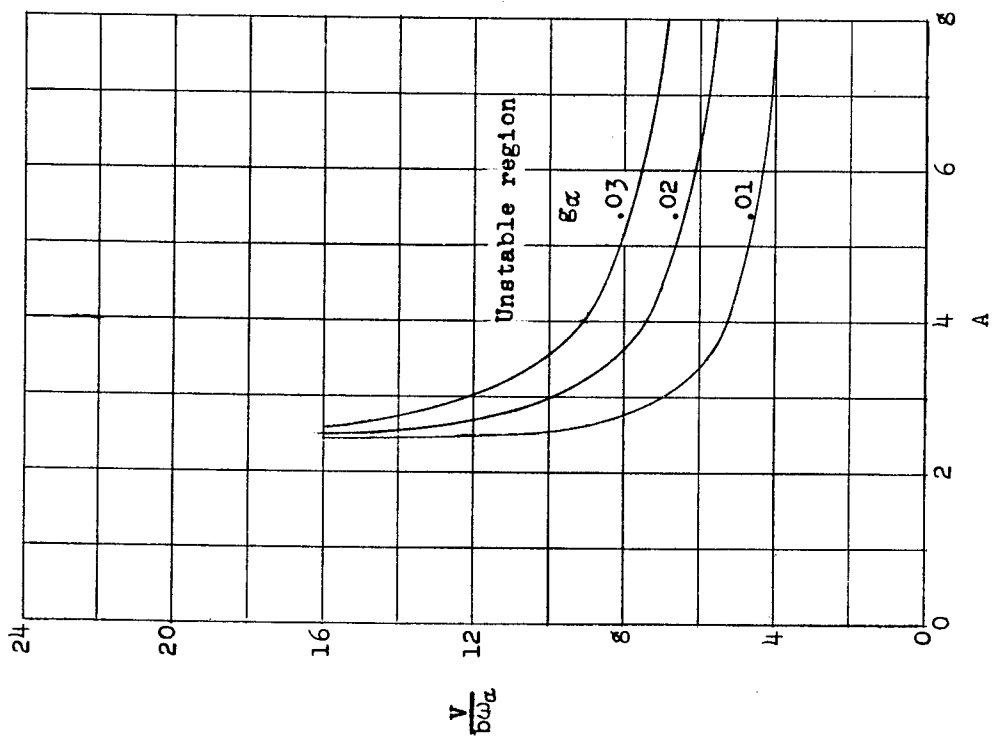
$$r_a^2 = 0.3.$$

$$r_a^2 = 0.3.$$

Figure 5.- Continued.



(j)  $M^2 = 1.8$ ;  $x_0 = 0.4$ ;  $\frac{1}{k} = 30$ ;  $\mu = 200$ ;  
 $r_a^2 = 0.3$ .



(i)  $M^2 = 1.4$ ;  $x_0 = 0.4$ ;  $\frac{1}{k} = 30$ ;  $\mu = 200$ ;  
 $r_a^2 = 0.3$ .

Figure 5.- Concluded.